# Calculating Areas and Centroids On the Sphere

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These functions are available in the ArcGIS extension "Tools for Graphics and Shapes"
*All formulae and references are described in full in the manual*

Available for free download at http://www.jennessent.com/arcgis/shapes_graphics.htm.

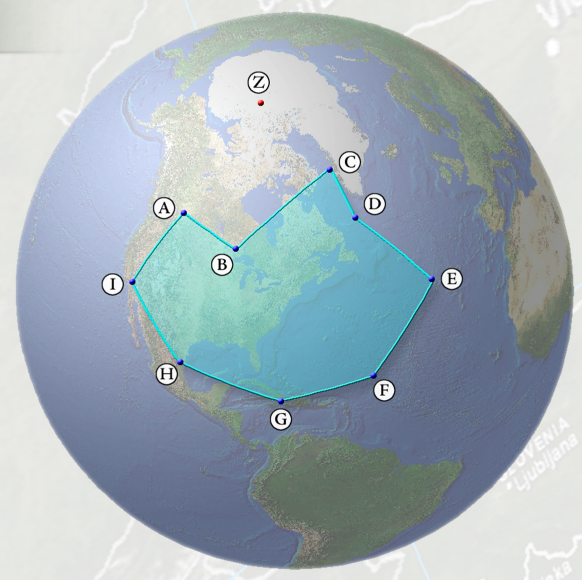
## Introduction

As we all know, projected data are not accurate. We cannot project features from the curved surface of the planet onto a flat surface while maintaining the original geometric attributes of those features. The best we can do is carefully select our projection based on how well that projection preserves area, shape, direction or distance. We must accept the fact that the projection can only maximize accuracy in one attribute by sacrificing accuracy in the other three.

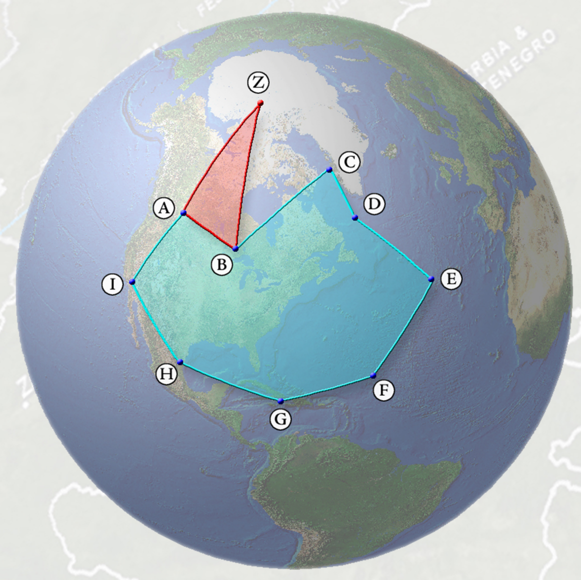
Too often, people do not give sufficient thought to the consequences of using a particular projection when calculating geometric attributes. Using the wrong projection can lead to dramatic errors in area, distance or centroid position. This poster illustrates a way to calculate area and centroid position directly on the sphere, and therefore provide more accurate, consistent and comparable data.

## Triangulate the Polygon

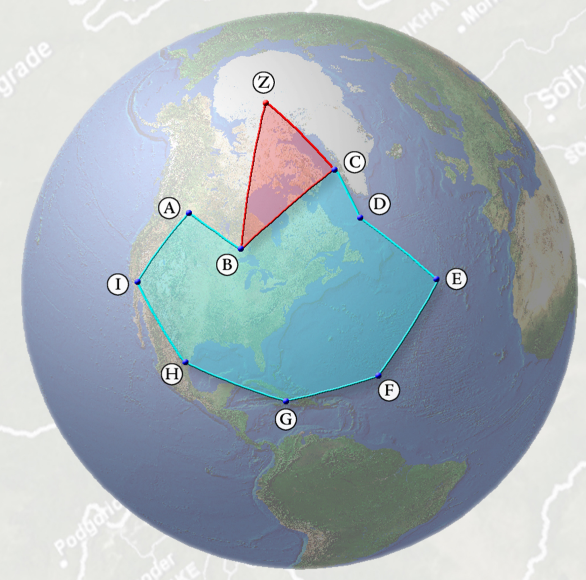
Decompose the polygon into a set of triangles, where each triangle is defined by a consecutive pair of vertices in the polygon plus an anchor point. This anchor point is constant for all triangles, and does not have to actually be inside the polygon boundary. Calculate the area of each triangle, assigning it a negative value if the triangle vertices are entered in a counter-clockwise direction. The polygon area is equal to the sum of the triangle areas, and the polygon centroid is equal to the area-weighted average of the triangle centroids.



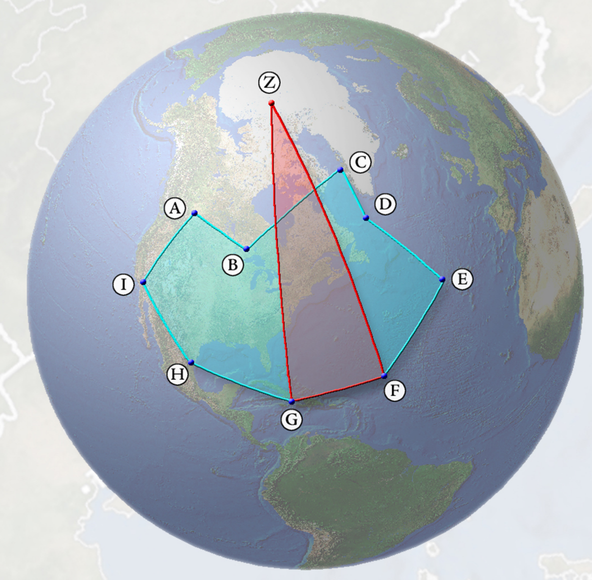
**Step 1:**
Set an anchor point. Note that this point does not have to be in the polygon



**Step 2:**
Generate triangles from consecutive pairs of vertices, plus the anchor point.



**Step 3:**
Note direction of the triangle vertices. The vertices for △BCZ above are entered in a counter-clockwise direction, so the area for △BCZ is set to negative.

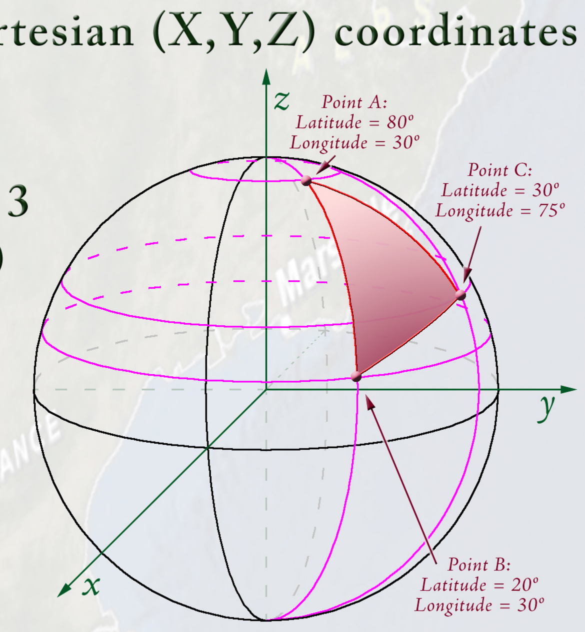


**Step 3b:**
The vertices for △FGZ are entered in a clockwise direction, so the area for △BCZ is set to positive.

## Triangle Centroid

Determining the surface center of mass of a spherical triangle is easy to do if you convert the Polar Latitude / Longitude coordinates to Cartesian (X,Y,Z) coordinates first:

Step 1: Given a triangle with 3 polar (Latitude / Longitude) coordinates:




Step 2: Convert polar coordinates to Cartesian X,Y,Z coordinates:

$\phi = \frac{(90 - \text{Latitude})\pi}{180}$ = Angle from North Pole down to Latitude, in Radians

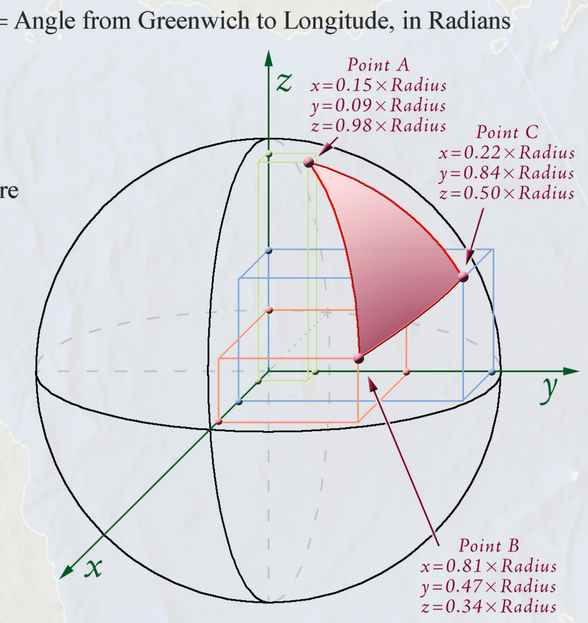
$\theta = \frac{(\text{Longitude})\pi}{180}$ = Angle from Greenwich to Longitude, in Radians

$X = R \sin\phi \cos\theta$

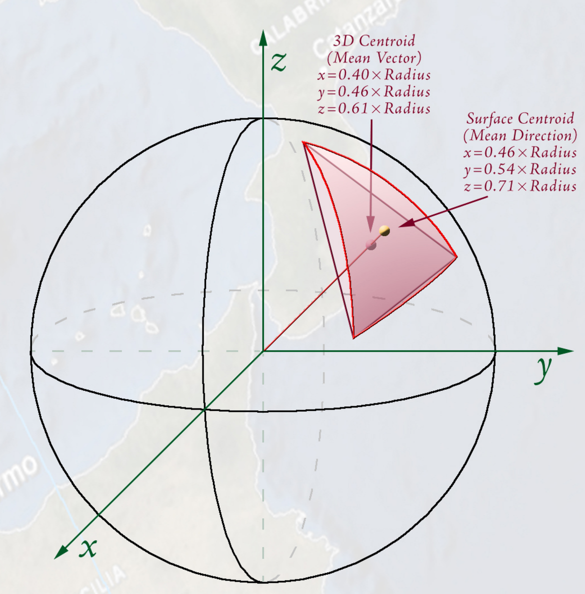
$Y = R \sin\phi \sin\theta$

$Z = R \cos\phi$

$R$ = Radius of Sphere




Step 3: Calculate the 3D centroid of the 3 triangle vertices, and then shift that point out to the surface of the planet.




3D Triangle Centroid:

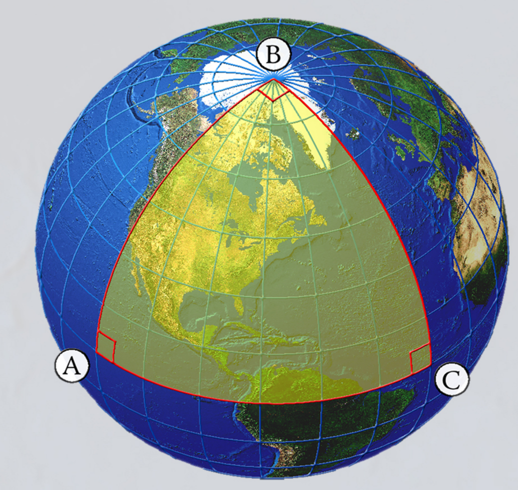
$$X_{3DC} = \frac{\sum(X_1 + X_2 + X_3)}{3} \qquad Y_{3DC} = \frac{\sum(Y_1 + Y_2 + Y_3)}{3} \qquad Z_{3DC} = \frac{\sum(Z_1 + Z_2 + Z_3)}{3}$$

Spherical Triangle Surface Centroid:

$$\text{Centroid Vector Length } = L = \sqrt{X_{3DC}^2 + Y_{3DC}^2 + Z_{3DC}^2}$$

$$\text{Surface X} = \frac{X_{3DC}}{L} \qquad \text{Surface Y} = \frac{Y_{3DC}}{L} \qquad \text{Surface Z} = \frac{Z_{3DC}}{L}$$

## Triangle Area




Triangle area can easily be calculated using either triangle side lengths or internal angles. Internal angles of a spherical triangle always exceed 180°, and this "spherical excess" is directly related to the triangle area.

$$\text{Area by Internal Angle} = R^2 E$$

where $R$ = sphere radius

$E$ = triangle spherical excess in radians

= sum of angles (in radians) - $\pi$

$$\text{Area by Edge Length} = R^2 E$$

$$\text{where } \tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{S}{2}\right)\tan\left(\frac{S-\overrightarrow{AB}}{2}\right)\tan\left(\frac{S-\overrightarrow{BC}}{2}\right)\tan\left(\frac{S-\overrightarrow{CA}}{2}\right)}$$

$\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ = Triangle Edge Lengths

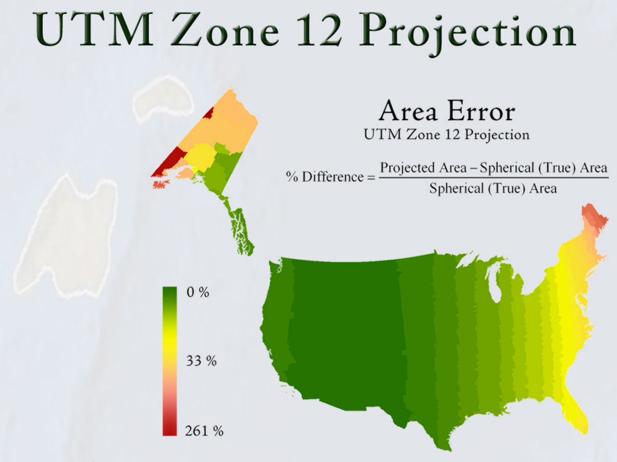
$$S = \frac{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}}{2} = \text{Triangle Half-Perimeter}$$

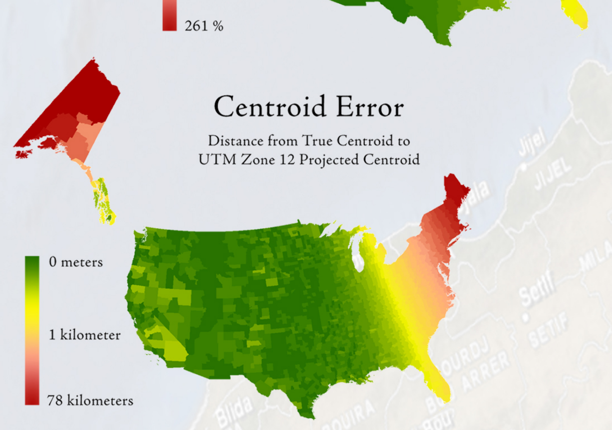
## An Examination of Errors Derived from Projected Data

I compared areas derived by spherical vs. projected methods by calculating the percent difference between projected and spherical area values, such that a value of 50% means that the projected area was 1.5 × as large as the spherical area, while a value of -50% means the projected area was half as large as the spherical area. I determined centroid accuracy by measuring the distance between spherical and projected centroids. I assume that the spherical values are the closest to truth and I describe the difference observed in the projected values as "error".

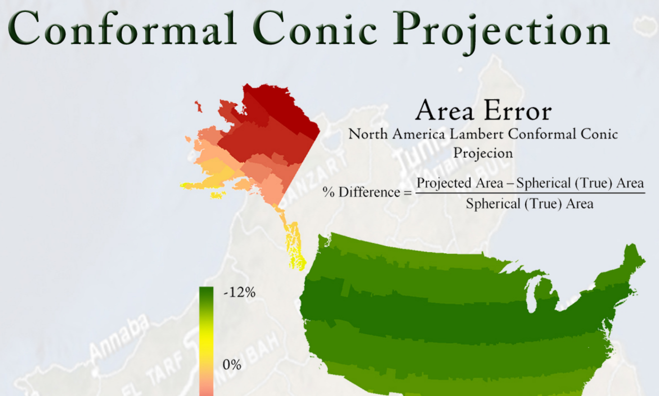
Interpretation of the results depends on your tolerance for error. Many projections do well over small areas, and all lose accuracy with increasing distance from the projection central meridian or latitude.

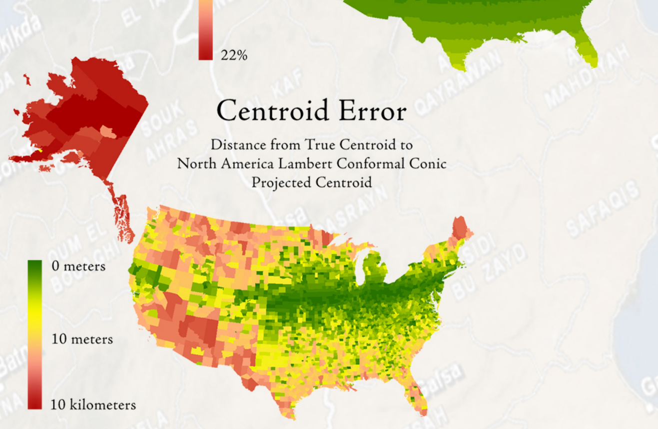
### UTM Zone 12 Projection




### North American Lambert Conformal Conic Projection




### North American Albers Equal Area Conic Projection